A Folded Square Sangaku Problem

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1 A sangaku problem

In Edo era, there was a unique mathematical custom in Japan. When people found nice problems, they wrote their problems on a framed wooden board, which was dedicated to a shrine or a temple. The board is called a sangaku (san (算) means mathematics and gaku (額) means framed board). It was also a mean to publish a discovery or to propose a problem. Most such problems were geometric and the figure were beautifully drawn in color (see the cover page). The uniqueness is not only the custom, but also the contents of the problems. Ordinary triangle geometry mainly concerns the properties of "one" triangle. On the contrary, sangaku problems concern about some relationship arising from a number of mixed elementary figures like circles, triangles, squares etc. For some nice examples of sangaku problems, see [3]. In this article, we consider a popular folded square problem (see Figure 1).

Problem. If a piece of square paper $ABCD$ is folded so that the corner D coincides with a point D' on the segment BC and the segment AD is carried into $A'D'$, which intersects AB at E, then the inradius of the triangle $BD'E$ equals $|A'E|$.

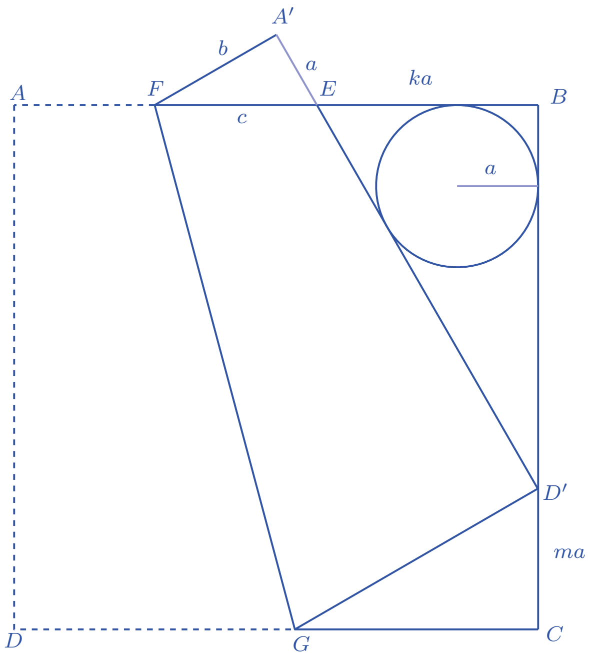

Figure 1: A sangaku problem

The figure has several interesting properties [1], [2] and [4]. Indeed six problems were proposed from this figure in [1]. Also it was used in the front cover of the journal on which [4] appeared. In this article we give one more property of the circumcircle of the same triangle, which seems to be new.

Solution. Let the crease intersect AB at F and $a = |A'E|$, $b = |A'F|$, $c = |EF|$ and $|BE| = ka$ for a real number k. Since the triangles $A'EF$ and BED' are similar, $|D'E| = kc$. From $|AB| = |A'D'|$, we get $b + c + ka = a + kc$. Therefore

$$k = \frac{a-b-c}{a-c} = \frac{(a-b-c)(a+c)}{a^2-c^2} = \frac{-b(a+c)-b^2}{-b^2} = \frac{a+b+c}{b}. \tag{1}$$

Since $BD'E$ is a right triangle, its inradius equals

$$\frac{ka+kb-kc}{2} = \frac{k(a+b-c)}{2} = \frac{(a+b)^2-c^2}{2b} = \frac{2ab}{2b} = a.$$

The solution is essentially the same as that of [2], but is slightly changed to show (1), which is needed later. We shall use the same notations throughout this article. By (1), $|BD'| = kb = a+b+c$ [1, problem 5], [4]. Let $|CD'| = ma$ for a real numbers m. $|AB| = a+kc = a+(a+b+c)c/b = (ab+ac+bc+c^2)/b = (a+c)(b+c)/b$ [4]. But $m(b+c) = |CD| = |AB|$. Therefore

$$m = \frac{a+c}{b}. \tag{2}$$

Let the crease intersect CD at G. By (2) $|CG| = a+c$ [4]. Also by (1) and (2)

$$k = m+1. \tag{3}$$

Hence $|A'E| + |CD'| = |BE|$ [1, problem 3], [4]. Also (3) implies that the sum of the perimeters of the triangles $A'FE$ and CGD' equals the perimeter of the triangle $BD'E$ [1, problem 4]. According to [1], this property was used in the 37th Slovenian Mathematical Olympiad in 1993.

2 Circumcircle

We now consider the circumcircle of the triangle $BD'E$ instead of the incircle (see Figure 2). Let X be the point of intersection of the lines AD and $A'D'$. The following theorem holds.

Theorem 1. *The circumcircle of the triangle $BD'E$ and the incircle of the quadrilateral $XDGD'$ are congruent.*

To prove the theorem, we need the following relation.

Proposition 1. *If $x^2+y^2=z^2$, then*

$$(x+y+z)^2 = 2(x+z)(y+z). \tag{4}$$

Proof. $(x+y+z)^2 - 2(x+z)(y+z) = x^2+y^2-z^2$. □

We now prove the theorem. The circumradius of the triangle $BD'E$ is $kc/2$. Let δ be the incircle of the quadrilateral, and let H and r be its center and radius, respectively. Let I be the point of tangency of δ and CD and let the incircle of the triangle $BD'E$ have center J and touch BC at K. If I' is the image of I by reflection in GH, then HI' ane $A'D'$ are parallel. Therefore $\angle JD'K = \frac{1}{2}\angle BD'E = \frac{1}{2}\angle IHI' = \angle GHI$. Hence the triangles JKD' and GIH are similar. Therefore we get $|KD'|/|JK| = |IH|/|GI|$, i.e., $(b+c)/a = r/(mc-r)$. Solving the equation for r with (1), (2) and (4), we get

$$r = \frac{c(b+c)m}{a+b+c} = \frac{c(b+c)(a+c)}{(a+b+c)b} = \frac{(a+b+c)c}{2b} = \frac{kc}{2}.$$

The theorem is proved.

We conclude this article with some remarks on Figure 2. Let T be the point of tangency of δ and $A'D'$. Then $HI'D'T$ is a square. Hence $|TD'|$ equals the circumradius, i.e., T is the center of the circumcircle, also the circumcircle passes through H. The triangle $D'HE$ is an isosceles right triangle with $|HD'| = |HE|$. The points D, D' and E lie on a circle of radius $kc/\sqrt{2}$ with center H.

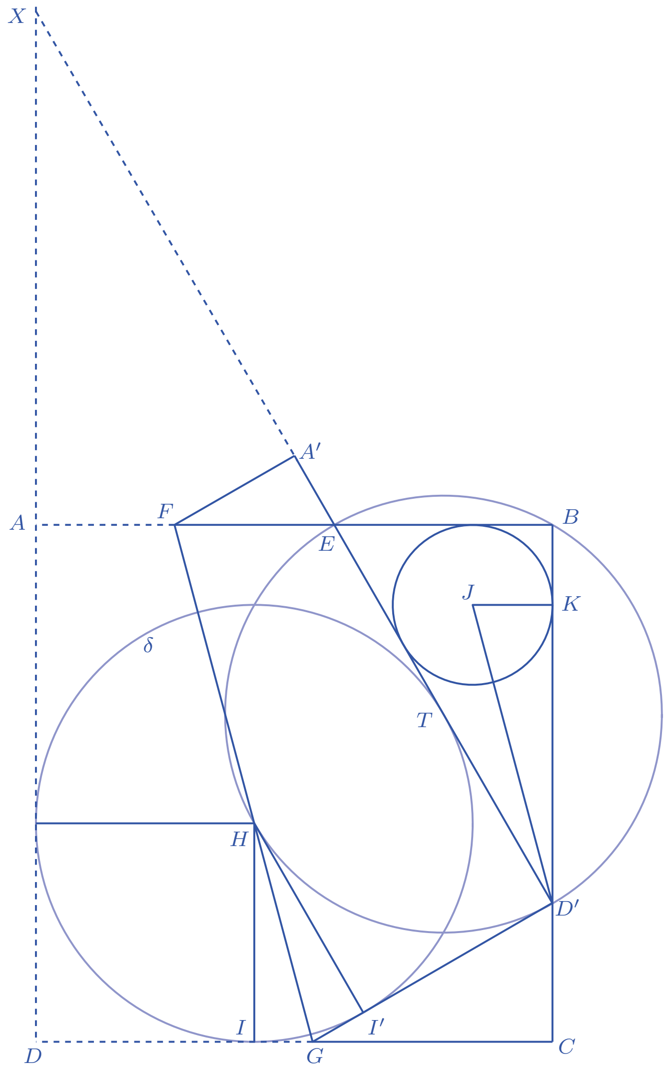

Figure 2: Circumcircle

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